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The Lomax generator of distributions: Properties, minification process and regression model

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ABSTRACT

We propose a new class of distributions called the Lomax generator with two extra positive parameters to generalize any continuous baseline distribution. Some special models such as the Lomax-normal, Lomax–Weibull, Lomax-log-logistic and Lomax–Pareto distributions are discussed. Some mathematical properties of the new generator including ordinary and incomplete moments, quantile and generating functions, mean and median deviations, distribution of the order statistics and some entropy measures are presented. We discuss the estimation of the model parameters by maximum likelihood. We propose a minification process based on the marginal Lomax-exponential distribution. We define a log-Lomax–Weibull regression model for censored data. The importance of the new generator is illustrated by means of three real data sets.

1. Introduction

In the past few decades, a major research effort has been devoted to the study of skew-symmetric distributions. Such distributions have been constructed by adding new parameters to a baseline cumulative density function (cdf) to obtain a new family of asymmetric distributions that are analytically more flexible. Zografos and Balakrishnan [34] and Ristić and Balakrishnan [26] proposed families of univariate distributions generated by gamma random variables. For any baseline cdf $G(x), x \in \mathbb{R}$, they defined the gamma-G generator (with an extra shape parameter $a > 0$) by the probability density function (pdf) and cdf

$$f(x) = \frac{1}{\Gamma(a)} \{-\log[1 - G(x)]\}^{a-1} g(x)$$

and

$$F(x) = \frac{\gamma(a, -\log[1 - G(x)])}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_0^{-\log[1-G(x)]} t^{a-1} e^{-t} dt,$$

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respectively, where $g(x) = dG(x)/dx, \Gamma(a) = \int_0^\infty t^{a-1}\,e^{-t}dt$ denotes the gamma function and $\gamma(a,z) = \int_0^z t^{a-1}\,e^{-t}dt$ denotes the incomplete gamma function.

In recent years there has been an increased interest in defining new generators for univariate continuous distributions by introducing one or more additional shape parameter(s) to the baseline distribution. This induction of parameter(s) has been proved useful in exploring tail properties and also for improving the goodness-of-fit of the proposed generator family. Some well-known generators are the beta-G by Eugene et al. [15] and Jones [19], Kumaraswamy-G by Cordeiro and de Castro [10], McDonald-G by Alexander et al. [1], Kummer beta-G by Pescim et al. [24], gamma-G by Zografos and Balakrishanan [34], Ristić and Balakrishanan [27] and Torabi and Montazari [32], log-gamma-G by Amini et al. [4], logistic-G by Torabi and Montazari [33], exponentiated generalized-G by Cordeiro et al. [12,13], Transformed-Transformer by Alzaatreh et al. [5], exponentiated Transformed-Transformer by Alzaghal et al. (2013), Weibull-G by Bourguignon et al. [6] and exponentiated half-logistic-G by Cordeiro et al. [14].

The aim of this note is to introduce a new family of distributions based on the Lomax distribution. The *Lomax-G* ("LG for short") generator with two extra positive parameters α and β is defined by the cdf and pdf given by

$$F(x) = \alpha\beta^\alpha \int_0^{-\log[1-G(x)]} \frac{dt}{(\beta+t)^{\alpha+1}} = 1 - \left\{\frac{\beta}{\beta - \log[1-G(x)]}\right\}^\alpha \tag{1}$$

and

$$f(x) = \alpha\,\beta^\alpha \frac{g(x)}{[1-G(x)]\,\{\beta - \log[1-G(x)]\}^{\alpha+1}}, \tag{2}$$

respectively, where $g(x) = dG(x)/dx$. The LG generator has the same parameters of the baseline G distribution plus two additional parameters α and β. The pdf (2) will be most tractable when $G(x)$ and $g(x)$ have simple analytic expressions. Henceforth, a random variable with density (2) is denoted by $X \sim \mathrm{LG}(\alpha,\beta)$.

The hazard rate function (hrf) of X (if the support is a positive real line) is given by

$$h(x) = \alpha \frac{g(x)}{[1-G(x)]\,\{\beta - \log[1-G(x)]\}}. \tag{3}$$

In the sequel, we consider the following expansions:

$$(1-z)^q = \sum_{k=0}^\infty (-1)^k \binom{q}{k} z^k, \quad \text{if } |z| < 1, \tag{4}$$

$$\log(1-z) = -z\sum_{k=0}^\infty \frac{z^k}{k+1}, \quad \text{if } |z| < 1 \tag{5}$$

and

$$\binom{-x}{n} = \frac{(-1)^n}{n!}\,x^{(n)}, \tag{6}$$

where $x^{(k)} = x(x+1)\ldots(x+k-1)$ denotes the rising factorial. We shall use the beta function $\mathrm{B}(a,b) = \int_0^1 x^{a-1}\,(1-x)^{b-1}\,dx$ and incomplete beta function $\mathrm{B}_x(a,b) = \int_0^x t^{a-1}\,(1-t)^{b-1}\,dt$.

The rest of the paper is organized as follows. In Section 2, we present some new generated distributions. A range of mathematical properties of (2) is derived in Sections 3 to 4. Maximum likelihood estimation of the model parameters and the observed information matrix are presented in Section 5. In Section 6, we introduce a minification process with marginal Lomax-exponential distribution, which has several important results. In Section 7, we define the *log-Lomax–Weibull* distribution and derive an expansion for its moments. Further, we propose a *log-Lomax–Weibull regression model*, estimate the parameters by the method of maximum likelihood and derive the observed information matrix. Three applications to real data are investigated in Section 8. Finally, some conclusions and future work are noted in Section 9.

2. Special LG distributions

The LG generator density function (2) allows for greater flexibility of its tails and can be widely applied in many areas of engineering and biology. Here, we introduce some special models obtained from this family because it extends several widely-known distributions in the literature. In the following examples, α and β are the Lomax generator parameters.

2.1. The Lomax-normal (LN) distribution

The LN distribution is defined from (2) by taking $G(x)$ and $g(x)$ to be the cdf and pdf of the normal $N(\mu,\sigma^2)$ distribution. The LN density function is given by

$$f_{LN}(x) = \alpha\beta^{\alpha}\,\frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\left\{\beta - \log\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha+1}\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]}, \tag{7}$$

where $x \in \mathbb{R}, \mu \in \mathbb{R}$ is a location parameter, $\sigma > 0$ is a scale parameter and $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution, respectively. A random variable with density (7) is denoted by $X \sim \mathrm{LN}(\alpha, \beta, \mu, \sigma^2)$. For $\mu = 0$ and $\sigma = 1$, we have the standard LN distribution.

2.2. The Lomax–Weibull (LW) distribution

Taking $G(x)$ to be the Weibull cdf with scale parameter $b > 0$ and shape parameter $a > 0$, say $G(x) = 1 - \exp[-(bx)^a]$, the LW density function (for $x > 0$) is given by

$$f_{LW}(x) = \alpha\,\beta^{\alpha}\,a\,b^{a}\,\frac{x^{a-1}}{[\beta + (xb)^{a}]^{\alpha+1}}. \tag{8}$$

For $a = 1$, we obtain the Lomax-exponential (LE) distribution. For $\beta = 1$ in addition to $b = 1$, it gives the two-parameter Burr distribution. For $\beta = \alpha = b = 1$, it becomes the one-parameter log-logistic distribution (also known as the one-parameter Fisk distribution in economics). Further, for $\beta = a = b = 1$, we obtain the Pareto type II distribution.

2.3. The Lomax–Pareto (LPa) distribution

Taking $G(x)$ to be a Pareto cdf with parameter $\nu > 0$, say $G(x) = 1 - (1 + x)^{-\nu}$, the LPa density function (for $x > 0$) becomes

$$f_{LP}(x) = \frac{\alpha\beta^{\alpha}\,\nu}{(1+x)\,[\beta + \nu\log(1+x)]^{\alpha+1}}. \tag{9}$$

2.4. The Lomax-log-logistic (LLL) distribution

The pdf and cdf of the log-logistic (LL) distribution are (for $x, a, b > 0$)

$$g(x) = \frac{b}{a^{b}}\,x^{b-1}\left[1 + \left(\frac{x}{a}\right)\right]^{-2} \quad \text{and} \quad G(x) = 1 - \left[1 + \left(\frac{x}{a}\right)^{b}\right]^{-1}.$$

Inserting these expressions into (2) gives the Lomax-log-logistic (LLL) density function (for $x > 0$)

$$f_{LLL}(x) = \frac{\alpha\beta^{\alpha}\,b}{a^{b}}\,\frac{x^{b-1}}{\left[1 + \left(\frac{x}{a}\right)^{b}\right]\left\{\beta + \log\left[1 + \left(\frac{x}{a}\right)^{b}\right]\right\}^{\alpha+1}}.$$

Fig. 1 displays some possible shapes of four generated pdf's. These plots show that the new distributions have great flexibility.

3. Main properties

In this section, we present two useful expansions for the cdf and pdf of X, describe analytically the shapes of pdf and hrf of the new family, obtain the quantile function (qf), ordinary and incomplete moments, probability weighted moments (PWMs) and generating function.

3.1. Useful expansions

From now on, we use an equation by Gradshteyn and Ryzhik [16, Section 0.314] for a power series raised to a positive integer n

$$\left(\sum_{i=0}^{\infty} a_i\,u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i}\,u^i, \tag{10}$$

where the coefficients $c_{n,i}$ (for $i = 1, 2, \ldots$) are easily determined from the recurrence equation

$$c_{n,i} = (i\,a_0)^{-1}\sum_{m=1}^{i}[m\,(n+1) - i]\,a_m\,c_{n,i-m}$$

and $c_{n,0} = a_0^n$.

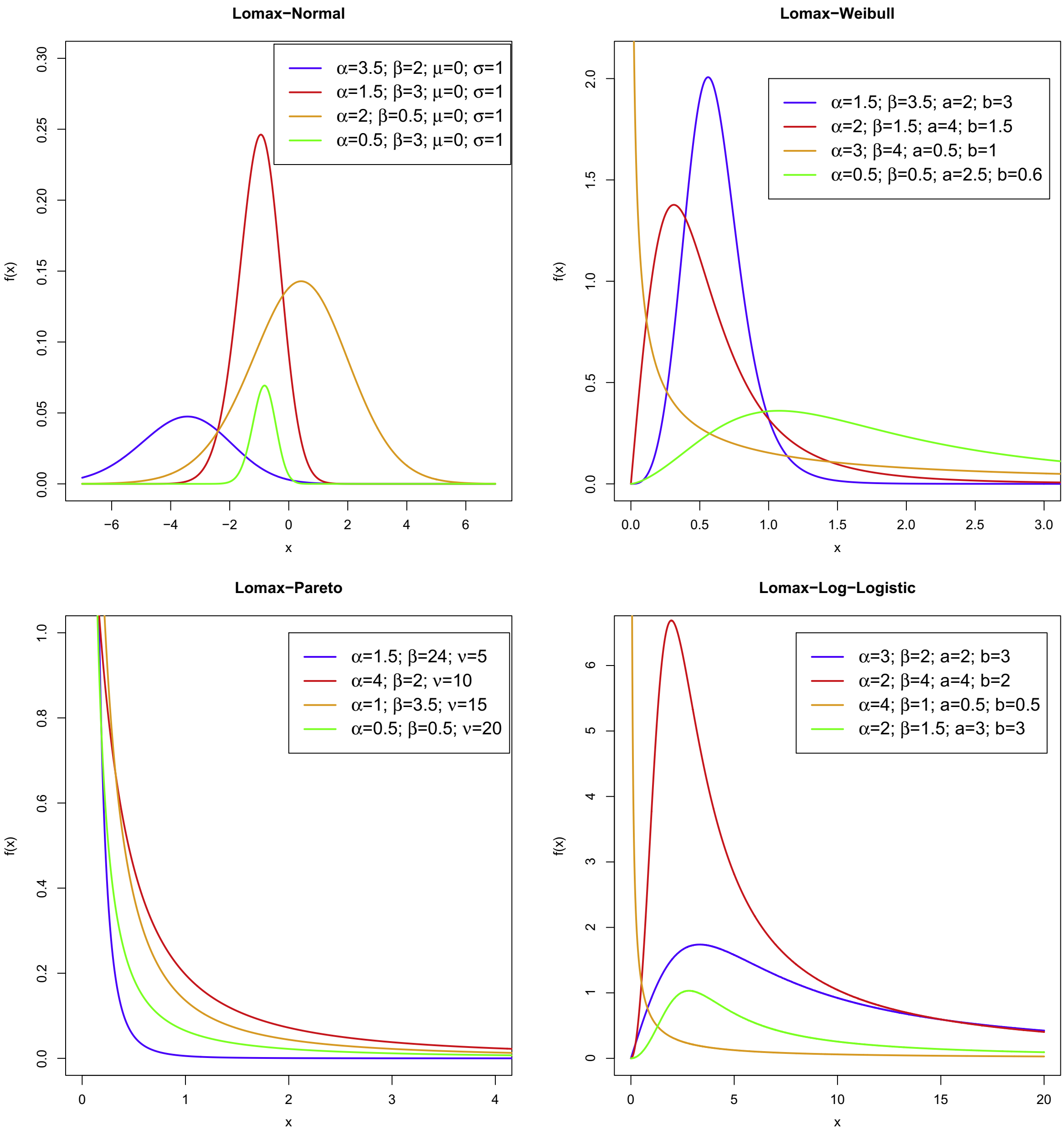

Fig. 1. Some possible shapes of the LG density functions.

The expansion of the cdf (1) can be expressed as

$$F(x) = \sum_{i,j \geq 0} v_{i,j} \, G^{i+j}(x), \tag{11}$$

where $v_{i,j} = (-1)^i d_{i,j} \, \alpha^{(i)} / (\beta^i \, i!)$ with $d_{i,0} = 1$ and (for $j \geq 1$) $d_{i,j} = j^{-1} \sum_{m=1}^{j} \frac{[m(i+1)-j]}{m+1} \, d_{i,j-m}$.

By simple differentiation, we can write

$$f(x) = g(x) \sum_{\substack{i,j \geq 0 \\ i+j \geq 1}} (i+j) \, v_{i,j} \, G^{i+j-1}(x). \tag{12}$$

Eqs. (11) and (12) are the main results of this section.

3.2. Shapes

The shapes of (2) and (3) can be described analytically. The critical points of the pdf of X are the roots of the equation:

$$\frac{g'(x)}{g(x)} - (\alpha+1)\frac{g(x)}{\{\beta - \log[1-G(x)]\}[1-G(x)]} + \frac{g(x)}{1-G(x)} = 0. \quad (13)$$

There may be more than one root to (13). If $x = x_0$ is a root of (13) then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether $\lambda(x_0) < 0, \lambda(x_0) > 0$ or $\lambda(x_0) = 0$, where

$$\lambda(x) = \frac{g''(x) - (g'(x))^2}{(g'(x))^2} - (\alpha+1)\frac{g'(x)\{\beta - \log[1-G(x)]\}[1-G(x)]}{\{\beta - [\log(1-G(x))]\}^2[1-G(x)]^2} + \frac{g'(x)[1-G(x)] + g^2(x)}{[1-G(x)]^2}.$$

The critical points of the hrf of X are the roots of the equation:

$$\frac{g'(x)}{g(x)} - \frac{g(x)}{\{\beta - \log[1-G(x)]\}[1-G(x)]} + \frac{g(x)}{1-G(x)} = 0. \quad (14)$$

There may be more than one root to (14). If $x = x_0$ is a root of (14) then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether $\tau(x_0) < 0, \tau(x_0) > 0$ or $\tau(x_0) = 0$, where

$$\tau(x) = \frac{g''(x) - (g'(x))^2}{(g'(x))^2} - \frac{g'(x)\{\beta - \log[1-G(x)]\}[1-G(x)]}{\{\beta - [\log(1-G(x))]\}^2[1-G(x)]^2} + \frac{g'(x)[1-G(x)] + g^2(x)}{[1-G(x)]^2}.$$

For the special LPa distribution, the expressions to obtain the critical points of the pdf and hrf are given by

$$\lambda(x) = \frac{(1+x)^{\nu+1}}{\nu} + \frac{\nu(\alpha+1)(\nu+1)}{(1+x)^2[\beta + \log(1+x)]} - \frac{\left[\nu^2 + (1+x)^2\right]}{(1+x)^2}$$

and

$$\tau(x) = \frac{(1+x)^{\nu+1}}{\nu} + \frac{\nu(\nu+1)}{(1+x)^2[\beta + \log(1+x)]} - \frac{\left[\nu^2 + (1+x)^2\right]}{(1+x)^2},$$

respectively.

3.3. Quantile function

By inverting (1), we obtain an explicit expression for the qf of the LG distribution as

$$Q(u) = F^{-1}(u) = Q_G\left(1 - e^{\beta\left[1-(1-u)^{-1/\alpha}\right]}\right), \quad (15)$$

where $Q_G(u) = G^{-1}(u)$ is the qf of the baseline G distribution and $u \in (0,1)$.

Using the power series for the exponential function and (4) and (10), Eq. (15) admits the power series

$$F^{-1}(u) = Q_G\left(1 - \sum_{k,m\geq 0}\sum_{l=0}^{m}\frac{(-1)^l\beta^k}{k!}\binom{k}{l}a_{l,m}\,u^m\right),$$

where $a_{l,0} = 1$ and $a_{l,m} = \frac{1}{m}\sum_{j=0}^{m}[j(l+1) - m]\binom{-1/\alpha}{m}(-1)^m a_{l,m-j}$.

Quantiles of interest can be obtained from (15) by substituting appropriate values for u. In particular, the median of X is given by

$$\text{Median}(X) = Q_G\left(1 - e^{\beta\left[1-0.5^{-1/\alpha}\right]}\right).$$

We can also use (15) for simulating LG variates: if U is a uniform random variable on the unit interval $(0,1)$, then

$$X = Q_G\left(1 - e^{\beta\left[1-(1-U)^{-1/\alpha}\right]}\right)$$

will be a LG random variable.

3.4. Moments

The ordinary moment $E(X^n)$ follows from (12) in terms of the baseline qf. We can write

$$\mu'_n = E(X^n) = \sum_{\substack{i,j\geq 0\\ i+j\geq 1}}(i+j)\,v_{i,j}\,\mathcal{I}_n(i,j), \quad (16)$$

where

$$\mathcal{I}_n(i,j)=\int_0^1 [Q_G(u)]^n\, u^{i+j-1}\,du. \qquad (17)$$

The ordinary moments of several LG distributions can be determined directly from Eqs. (16) and (17). Here, we provide three examples. For the LE distribution, using (4) and a result from [25, Section 2.6.3, Eq. (1)], we obtain

$$\mathcal{I}_n(i,j)=\frac{n!}{\lambda^n}\sum_{l=0}^{i+j-1}\frac{(-1)^l}{(l+1)^{n+1}}\binom{i+j-1}{l}.$$

For the LW model, using Eq. (1) in [25, Section 2.6.3], the baseline integral (17) turns out to be

$$\mathcal{I}_n(i,j)=\frac{\Gamma(n/a+1)}{b^n}\sum_{l=0}^{i+j-1}\frac{(-1)^l}{(l+1)^{n/a+1}}\binom{i+j-1}{l}. \qquad (18)$$

For the LPa distribution, we have

$$\mathcal{I}_n(i,j)=\sum_{l=0}^{n}\sum_{q\geq 0}\frac{(-1)^{n+l}\,(l/\nu)^{(q)}}{(i+j+q)\,q!}\binom{n}{l}.$$

The first four moments of the LW distribution are calculated by numerical integration and through infinite weighted sums in Eqs. (16) and (18) using the statistical software R. The values from both techniques are usually close when ∞ is replaced by a large number as 500 in (16) and (18). For specified values $\alpha=2.0, \beta=1.5, a=3.5$ and $b=1.5$, Table 1 gives some numerical analysis for these moments and for variance, skewness and kurtosis.

For the LN distribution, plots of the skewness and kurtosis for some values of β as functions of α are displayed in Fig. 2. For the LW distribution, plots of the skewness and kurtosis for some values of β as functions of α are displayed in Fig. 3.

For empirical purposes, the shape of many distributions can be usefully described by what we call the incomplete moments. These types of moments play an important role for measuring inequality, for example, income quantiles and Lorenz and Bonferroni curves, which depend upon the incomplete moments of a distribution. The nth incomplete moment of X can be determined as

$$m_n(y)=\sum_{\substack{i,j\geq 0\\ i+j\geq 1}}(i+j)\,v_{i,j}\int_0^{G(y)} Q_G(u)^n\,u^{i+j-1}\,du. \qquad (19)$$

The last integral can be computed for most baseline G distributions.

Further, the central moments (μ_r) and cumulants (κ_r) of X can be calculated as

$$\mu_r=\sum_{k=0}^{r}(-1)^k\binom{r}{k}\mu_1'^k\,\mu_{r-k}' \quad \text{and} \quad \kappa_r=\mu_r'-\sum_{k=1}^{r-1}\binom{r-1}{k-1}\kappa_k\,\mu_{r-k}',$$

respectively, where $\kappa_1=\mu_1'$. The skewness $\gamma_1=\kappa_3/\kappa_2^{3/2}$ and kurtosis $\gamma_2=\kappa_4/\kappa_2^2$ are computed from the second, third and fourth cumulants.

The nth descending factorial moment of X is

$$\mu_{(n)}'=E(X^{(r)})=E[X(X-1)\times\cdots\times(X-r+1)]=\sum_{k=0}^{r}s(r,k)\,\mu_k',$$

Table 1
First four moments, variance, skewness and kurtosis of the LW distribution for $\alpha=2.0, \beta=1.5, a=3.5$ and $b=1.5$ obtained by numerical integration from infinite weighted sums, where $i,j,l=0,\ldots,p$.

Moments	Infinite weighted sums				Numerical integration
	$p=50$	$p=100$	$p=200$	$p=500$	
μ_1'	0.61408	0.61392	0.61387	0.61385	0.61384
μ_2'	0.44296	0.44248	0.44229	0.44221	0.44218
μ_3'	0.37292	0.37235	0.37209	0.37194	0.37187
μ_4'	0.36734	0.36902	0.36996	0.37061	0.37115
Variance	0.06586	0.06558	0.06545	0.06539	0.06537
Skewness	140.01370	143.13160	144.15410	144.46400	144.43200
Kurtosis	6.21567	6.76574	7.07688	7.28331	7.44550

where

$$s(r,k) = (k!)^{-1}\left[\frac{d^k}{dx^k}x^{(r)}\right]_{x=0}$$

is the Stirling number of the first kind which counts the number of ways to permute a list of r items into k cycles. So, we can obtain the factorial moments from the ordinary moments given before.

Now, we obtain the probability weighted moments (PWMs) of X. They cover the summarization and description of theoretical probability distributions. The primary use of these moments is in the estimation of parameters for a distribution whose inverse cannot be expressed explicitly. The $(s,r)th$ PWM of X is formally defined as

$$\tau_{s,r} = E\{X^s F(X)^r\} = \int_{-\infty}^{\infty} x^s F(x)^r f(x)\,dx.$$

First, we obtain an expansion for $F(x)^r$. We can rewrite (11) as

$$F(x) = \sum_{p\geq 0} t_p\, G^p(x),$$

where $t_p = \sum_{i,j\geq 0; i+j=p} v_{i,j}$ for $p = 0,1,\ldots$. Further, using (10), we can write

$$F(x)^r = \left(\sum_{p\geq 0} t_p\, G^p(x)\right)^r = \sum_{p\geq 0} w_{r,p}\, G^p(x), \tag{20}$$

where $w_{r,0} = t_0^r$ and (for $p \geq 1$) $w_{r,p} = (p\,t_0)^{-1}\sum_{m=1}^{p}[m\,(r+1) - p]\,t_m\,w_{r,p-m}$ is determined recursively.

Second, we obtain from (12) by setting $G(x) = u$

$$\tau_{s,r} = \sum_{p\geq 0}\sum_{\substack{i,j\geq 0\\ i+j\geq 1}} (i+j)\, v_{i,j}\, w_{r,p}\, \mathcal{K}_s(p+i+j-1), \tag{21}$$

where $\mathcal{K}_s(q) = \int_0^1 Q_G(u)^s\, u^q\, du$ can be determined numerically from any baseline qf. Usually, we can derive closed-form expressions for (21) as shown in the following examples. First, for the LE (with parameter $\lambda > 0$) distribution, we obtain $\mathcal{K}_s(q) = s!\,\lambda^s \sum_{j=0}^{\infty} \frac{(-1)^{s+j}}{(j+1)^{n+1}} \binom{q}{j}$. Second, for the LSL, where $G(x) = (1+e^{-x})^{-1}$, we have $\mathcal{K}_s(q) = \left(\frac{\partial}{\partial z}\right)^s B(z+q+1, 1-z)|_{z=0}$. Third, for the LW distribution, where $G(x) = 1 - e^{-(bx)^a}$, we obtain $\mathcal{K}_s(q) = a^{-s}\,\Gamma(s/b+1)\sum_{i=0}^{\infty}\frac{(-q)_i}{i!\,(i+1)^{(s+b)/b}}$, where $(-q)_i = (-q)(-q-1)\ldots(-q-i+1)$ is the falling factorial.

The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. For the LN distribution, plots of the skewness and kurtosis for selected values of β as functions of α are displayed in Fig. 2. For the LW distribution, plots of the skewness and kurtosis for some values of β as functions of α are displayed in Fig. 3.

Other kinds of moments such L-moments may also be obtained in closed-form, but we consider only the previous moments for reasons of space.

3.5. Generating function

Here, we provide a simple formula for the moment generating function (mgf) $M(t) = E(e^{tX})$ of X from Eq. (12) as

$$M(t) = \sum_{\substack{i,j\geq 0\\ i+j\geq 1}} (i+j)\, v_{i,j}\, \mathcal{J}(t;i,j), \tag{22}$$

where

$$\mathcal{J}(t;i,j) = \int_{-\infty}^{+\infty} e^{tx}\, g(x)\, G^{i+j-1}(x)\,dx$$

can be determined for most baseline qf's as

$$\mathcal{J}(t;i,j) = \int_0^1 \exp[t\,Q_G(u)]\, u^{i+j-1}\,du. \tag{23}$$

For example, the integral (23) for the LE, LW and LPa distributions are given by

$$\mathcal{J}(t;i,j) = \sum_{l\geq 0}\frac{1}{(l+i+j)\,l!}\left(\frac{t}{\lambda}\right)^{(l)},$$

$$\mathcal{J}(t;i,j) = \sum_{l\geq 0}\frac{1}{(l+i+j)\,l!}\left(\frac{ta}{b}\right)^{(l)}$$

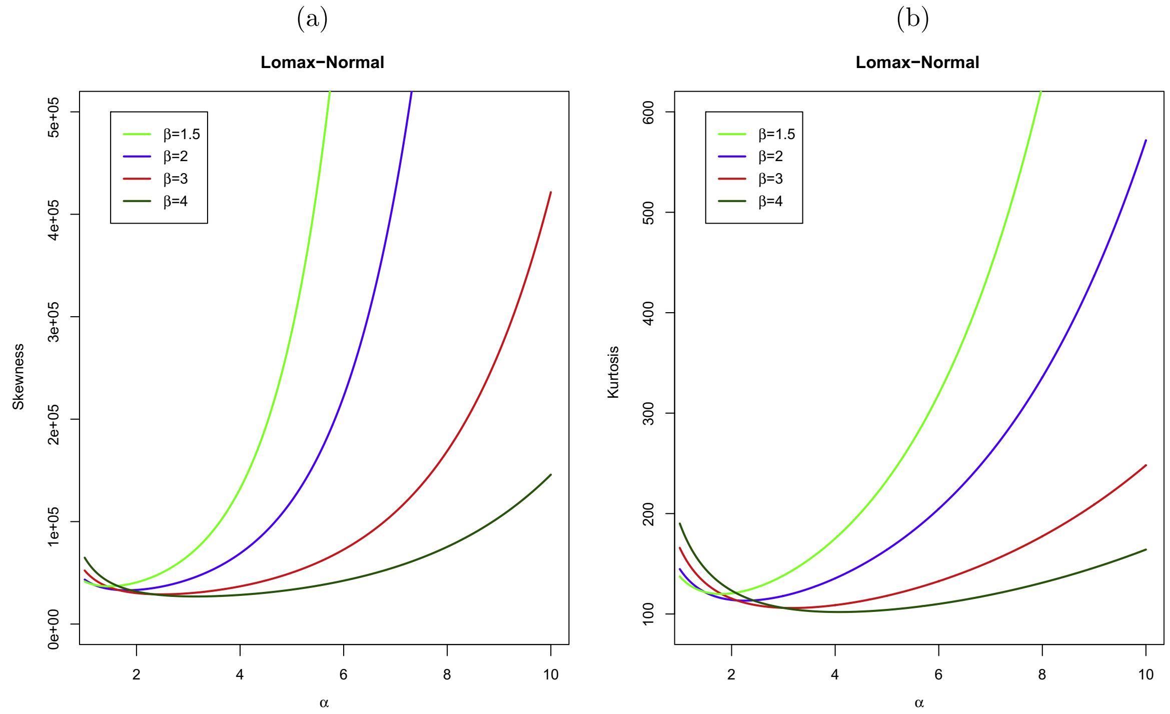

Fig. 2. For the LN distribution. (a) Skewness of X as a function of α for some values of β by taking $\mu = 0$ and $\sigma = 1$. (b) Kurtosis of X as a function of α for some values of β by taking $\mu = 0$ and $\sigma = 1$.

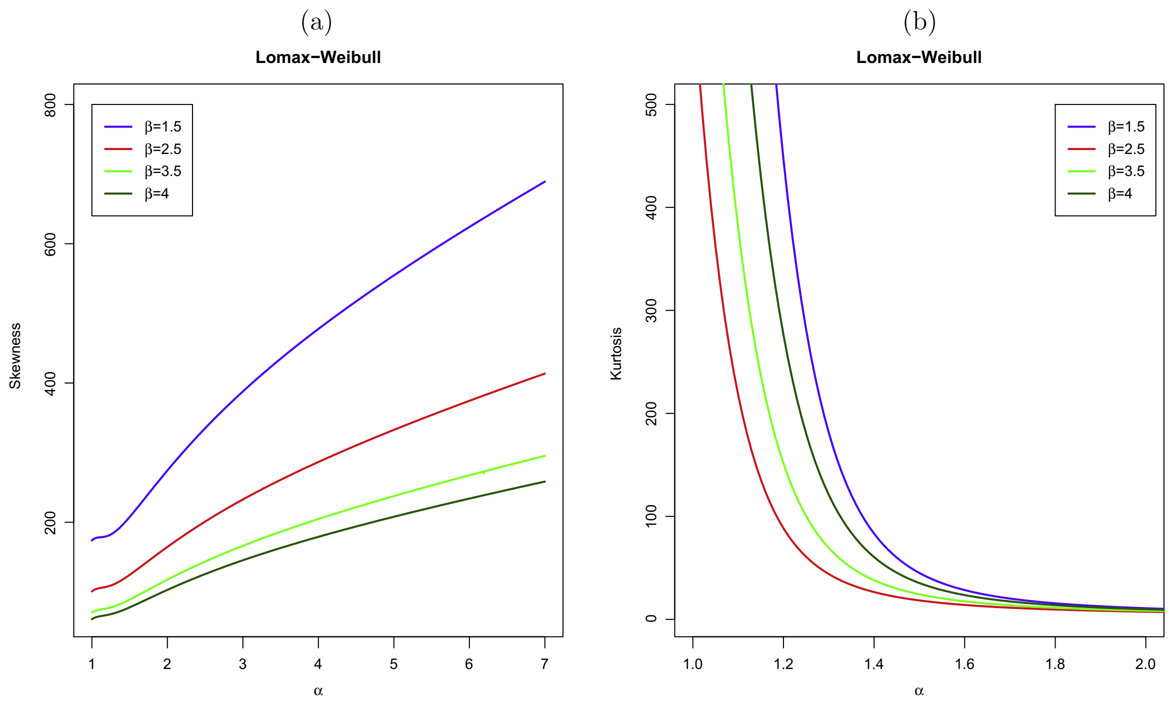

Fig. 3. For the LW distribution. (a) Skewness of X as a function of α for some values of β by taking $a = 3$ and $b = 2$. (b) Kurtosis of X as a function of α for some values of β by taking $a = 3$ and $b = 2$.

and

$$\mathcal{J}(t;i,j)=\sum_{l,q\geqslant 0}\sum_{m=0}^{l}(-1)^{l+m}\frac{t^{l}(m/v)^{(q)}}{(i+j+q)\,l!\,q!}\binom{l}{m},$$

respectively.

4. Other measures

In this section, we calculate the following measures: means deviations, order statistics, entropies, reliability for the LG distributions.

4.1. Mean deviations

The mean deviations about the mean ($\delta_1=E(|X-\mu_1'|)$) and about the median ($\delta_2=E(|X-M|)$) of X can be expressed as

$$\delta_1=2\mu_1'F(\mu_1')-2m_1\left(\mu_1'\right)\quad\text{and}\quad\delta_2=\mu_1'-2m_1(M),\tag{24}$$

respectively, where $\mu_1'=E(X), M=Median(X)$ is the median given in Section 3.3, $F(\mu_1')$ is easily calculated from the cdf (1) and $m_1(z)=\int_{-\infty}^{z}xf(x)dx$ is the first incomplete moment given by (19). The mean deviations of several LG distributions can be calculated directly from (24). Here, we provide three examples. For the LE distribution, we obtain from (10)

$$m_1(z)=\frac{1}{\lambda}\sum_{\substack{i,j,l\geq 0\\ i+j\geq 1}}(i+j)\,v_{i,j}\,\frac{\left(1-e^{-\lambda z}\right)^{l+i+j+1}}{(l+1)\,(l+i+j+1)}.\tag{25}$$

Based on Eqs. (4)–(6), we obtain the first incomplete moment of the LW distribution as

$$m_1(z)=\frac{1}{b}\sum_{\substack{i,j,l,p\geq 0\\ i+j\geq 1}}\sum_{m=0}^{l}(-1)^{i+l+m}\binom{1/a}{l}\binom{l}{m}\frac{(i+j)\,v_{i,j}\,d_{m,p}}{(p+m+i+j)}\times\left(1-e^{-b^{a}z^{a}}\right)^{p+m+i+j}.\tag{26}$$

The first incomplete moment of the LPa distribution for $v>1$ becomes

$$m_1(z)=\sum_{\substack{i,j\geq 0\\ i+j\geq 1}}(i+j)\,v_{i,j}\left\{B_{1-(1-z)^{-v}}\left(i+j,1-\frac{1}{v}\right)-\frac{\left[1-(1-z)^{-v}\right]^{i+j}}{i+j}\right\}.\tag{27}$$

So, the mean deviations for these three distributions can be obtained from Eqs. (24) and (25)–(27).

Applications of these equations can be addressed to obtain Bonferroni and Lorenz curves defined for a given probability π by $B(\pi)=m_1(q)/(\pi\mu_1')$ and $L(\pi)=m_1(q)/\mu_1'$, respectively, where $\mu_1'=E(X)$, $q=Q_G(1-e^{\beta[1-(1-\pi)^{-1/\alpha}]})$ is the LG qf given by (15) at π and $m_1(q)$ is given by (19) with $n=1$.

4.2. Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose $X_1,\ldots,X_n$ is a random sample from the Lomax-G distribution. Let $X_{i:n}$ denote the ith order statistic. From Eqs. (12) and (20), the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x)=K\,g(x)\sum_{j=0}^{n-i}\sum_{\substack{l,m,p\geq 0\\ l+m\geq 1}}(-1)^{j}\,(l+m)\,v_{l,m}\binom{n-i}{j}G^{p+l+m-1}(x),\tag{28}$$

where $K=n!/[(i-1)!\,(n-i)!]$.

Here, we derive two formulae for moments of the order statistics. The first one follows directly from its definition. Assuming that the moments of $X_{i:n}$ exist, we can write from (28)

$$E(X_{i:n}^{k})=\int_{-\infty}^{+\infty}x^{k}f_{i:n}(x)\,dx=K\sum_{j=0}^{n-i}\sum_{\substack{l,m,p\geq 0\\ l+m\geq 1}}(-1)^{j}\,(l+m)\,v_{l,m}\binom{n-i}{j}G^{p+l+m-1}(x)\,\mathcal{L}(p,l,m),\tag{29}$$

where $\mathcal{L}(p,l,m)=\int_0^1\left[G^{-1}(u)\right]^{k}u^{p+l+m-1}\,du$.

Using similar algebra as for the ordinary moments, the integral $\mathcal{L}(p,l,m)$ for the LE, LW and LPa distributions can be expressed as

$$\mathcal{L}(p,l,m)=\frac{k!}{\lambda^k}\sum_{l=0}^{p+l+m-1}\frac{(-1)^l}{(l+1)^{k+1}}\binom{p+l+m-1}{l},$$

$$\mathcal{L}(p,l,m)=\frac{\Gamma\left(\frac{k}{a}+1\right)}{b^k}\sum_{l=0}^{p+l+m-1}\frac{(-1)^l}{(l+1)^{k/a+1}}\binom{p+l+m-1}{l}$$

and

$$\mathcal{L}(p,l,m)=\sum_{l=0}^{k}\sum_{q\geq 0}\frac{(-1)^{k+l}(k/v)^{(q)}}{(p+l+m+q)\,q!}\binom{k}{l},$$

respectively.

4.3. Entropies

An entropy is a measure of variation or uncertainty of a random variable X. Two popular entropy measures are the Rényi and Shannon entropies [29]; 28). The Rényi entropy of a random variable with pdf $f(\cdot)$ is defined as

$$I_R(\gamma)=\frac{1}{1-\gamma}\log\int_0^\infty f^\gamma(x)dx$$

for $\gamma>0$ and $\gamma\neq 1$. The Shannon entropy of a random variable X is defined by $E\{-\log[f(X)]\}$. It is the particular case of the Rényi entropy when $\gamma\uparrow 1$.

Here, we derive expressions for the Rényi and Shannon entropies for the LG distribution. Due to the fact that the parameter γ is not in general a natural number it is difficult to use (12) for entropy derivation. So, we use (2), (4), (5) and (10) to obtain the Rényi entropy of X as

$$I_R(\gamma)=\frac{1}{1-\gamma}\left\{\gamma\log\alpha-\gamma\log\beta+\log\left[\sum_{i,j,k\geq 0}\frac{(-1)^i[\gamma(\alpha+1)]^{(i)}\,\gamma^{(j)}\,d_{i,k}}{\beta^i\,i!\,j!}\;\mathcal{D}(\gamma,\,i,\,j,\,k)\right]\right\},\qquad(30)$$

where $\mathcal{D}(\gamma,\,i,\,j,\,k)=\int_0^1\left[g\left(G^{-1}(u)\right)\right]^{\gamma-1}u^{i+j+k}\,du$. In practice, it is enough to calculate this integral. For example, for the LE, LW and LPa distributions, the integral turns out to be

$$\mathcal{D}(\gamma,\,i,\,j,\,k)=\lambda^{\gamma-1}\,\mathrm{B}(i+j+k+1,\gamma),$$

$$\mathcal{D}(\gamma,\,i,\,j,\,k)=b^{\gamma-1}\,\Gamma\left(\frac{(a-1)(\gamma-1)}{a}+1\right)\sum_{p=0}^{i+j+k}(-1)^p(p+\gamma)^{-\frac{(a-1)(\gamma-1)}{a}-1}\binom{i+j+k}{p},$$

where $a>\frac{\gamma-1}{\gamma}$, and

$$\mathcal{D}(\gamma,\,i,\,j,\,k)=v^{\gamma-1}\,\mathrm{B}\left(i+j+k+1,\frac{v+1}{v}+1\right),$$

respectively.

The Shannon entropy can be obtained by limiting $\gamma\uparrow 1$ in (30). However, it is easier to derive an expression for it from first principles. We can write

$$E\{-\log[f(X)]\}=-\log\alpha-\alpha\log\beta+E\{-\log[g(X)]\}+(\alpha+1)E\{\log[\beta-\log(1-G(X))]\}+E[\log(1-G(X))].\qquad(31)$$

Based on Eqs. (5), (12) and a result due to [25, Section 2.6.3, Eq. (1)], we can write

$$E\{-\log[g(X)]\}=-\sum_{\substack{i,j\geq 0\\ i+j\geq 1}}(i+j)\,v_{i,j}\,\mathcal{H}_1(i,j),\qquad(32)$$

$$E\{\log[\beta-\log(1-G(X))]\}=-\sum_{\substack{i,j\geq 0\\ i+j\geq 1}}(i+j)\,v_{i,j}\,\mathcal{H}_2(i,j)\qquad(33)$$

and

$$E\{\log[1-G(X)]\}=-\sum_{\substack{i,j\geq 0\\ i+j\geq 1}}(i+j)\,v_{i,j}\,\mathcal{H}_3(i,j),\qquad(34)$$

where

$$\mathcal{H}_1(i,j) = \int_0^1 \log\left[g\left(G^{-1}(u)\right)\right] u^{i+j-1}\,\mathrm{d}u, \tag{35}$$

$$\mathcal{H}_2(i,j) = \frac{\log b}{i+j} + \sum_{l\geq 0}\sum_{p=0}^{i+j-1} \frac{(-1)^p\,(l+1)!}{\beta^{l+1}\,(l+1)\,(p+1)^{l+2}} \binom{i+j-1}{p} \tag{36}$$

and

$$\mathcal{H}_3(i,j) = -\sum_{l\geq 0} \frac{1}{(l+1)(l+i+j+1)}. \tag{37}$$

So, from Eqs. (32)–(37) one can obtain expressions for the LG Shannon entropy.

The integral (35) can be determined for most baseline distributions using a power series expansion for $Q_G(u)$. In all forthcoming examples in this section, we use a result in [25, Section 2.6.3, Eq. (1)]. For the LE distribution, we have

$$\mathcal{H}_1(i,j) = \frac{\log\lambda}{i+j} - \sum_{p=0}^{i+j-1} \frac{(-1)^p}{(p+1)^2}\binom{i+j-1}{p}.$$

To calculate the integral (35) for the LW distribution, we use equation 4.331.1 in [16]. So,

$$\mathcal{H}_1(i,j) = \frac{\log\beta}{i+j} - \frac{a-1}{a}\sum_{l=0}^{i+j-1} \frac{(-1)^l\,[\mathbf{C} + \log(l+1)]}{(l+1)}\binom{i+j-1}{l} - \sum_{p=0}^{i+j-1}\frac{(-1)^p}{(p+1)^2}\binom{i+j-1}{l},$$

where $\mathbf{C} = 0.577215\ldots$ is the Euler's constant.

For the LPa distribution, the integral (35) reduces to

$$\mathcal{H}_1(i,j) = \frac{\log\nu}{i+j} - \frac{\nu+1}{\nu}\sum_{l=0}^{i+j-1}\frac{(-1)^l}{(l+1)^2}\binom{i+j-1}{l}.$$

Now, it is easy to obtain the LG Shannon entropy given by (31).

4.4. Reliability

Here, we derive the reliability $R = \Pr(X_2 < X_1)$, when $X_1 \sim$Lomax-$G(\alpha_1, \beta_1, \tau)$ and $X_2 \sim$ Lomax-$G(\alpha_2, \beta_2, \tau)$ are independent random variables and τ is the vector of parameters of the baseline G distribution. Probabilities of this form have many applications especially in engineering concepts. Let f_i denote the pdf of X_i and F_i denote the cdf of X_i. By using 4,5,6,(10), the mixture representations (11), (12), and a result from [25, Section 2.6.3, Eq. (1)], we obtain

$$R = \int_0^{+\infty} f_1(x)F_2(x)\,\mathrm{d}x = \alpha_1 \sum_{i,j,k\geq 0}\sum_{l=0}^{k} \frac{(-1)^{i+j+l}\,(\alpha_1+1)^{(i)}\,\alpha_2^{(j)}\,\Gamma(i+j+1)}{\beta_1^{i+1}\,\beta_2^{j}\,(k+1)^{i+j+1}\,i!\,j!}\binom{k}{l}. \tag{38}$$

In the particular case, $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, Eq. (38) reduces to $R = 1/2$.

5. Maximum likelihood estimation

Here, we consider estimation of the unknown parameters of the LG distribution by the maximum likelihood method. Let $x_1, \ldots, x_n$ be a sample from (2). Let $\mathbf{\Theta}$ be a $q\times 1$ parameter vector in $G(\cdot)$. The log-likelihood function $\log L = \log L(\alpha, \beta, \mathbf{\Theta})$ is

$$\log L = n\log\alpha + \alpha n\log\beta + \sum_{i=1}^{n}\log g(x_i) - (\alpha+1)\sum_{i=1}^{n}\log\{\beta - \log[1-G(x_i)]\} - \sum_{i=1}^{n}\log[1-G(x_i)]. \tag{39}$$

The first derivatives of $\log L$ with respect to the parameters α, β and $\mathbf{\Theta}$ are:

$$\frac{\partial\log L}{\partial\alpha} = \frac{n}{\alpha} + n\log\beta - \sum_{i=1}^{n}\log\{\beta - \log[1-G(x_i)]\},$$

$$\frac{\partial\log L}{\partial\beta} = \frac{\alpha n}{\beta} - (\alpha+1)\sum_{i=1}^{n}\frac{1}{\beta - \log[1-G(x_i)]},$$

$$\frac{\partial\log L}{\partial\mathbf{\Theta}} = \sum_{i=1}^{n}\frac{\dot g(x_i)}{g(x_i)} - (\alpha+1)\sum_{i=1}^{n}\frac{\dot G(x_i)}{\{\beta - \log[1-G(x_i)]\}\{1-G(x_i)\}} + \sum_{i=0}^{n}\frac{\dot G(x_i)}{1-G(x_i)},$$

where $\dot g(x_i) = \partial g(x_i)/\partial\theta$ and $\dot G(x_i) = \partial G(x_i)/\partial\theta$. The maximum likelihood estimates (MLEs) of $(\alpha, \beta, \mathbf{\Theta})$, say $(\widehat{\alpha}, \widehat{\beta}, \widehat{\mathbf{\Theta}})$, are the simultaneous solutions of the equations $\partial\log L/\partial\alpha = 0$, $\partial\log L/\partial\beta = 0$ and $\partial\log L/\partial\mathbf{\Theta} = \mathbf{0}$.

Maximization of (39) can be performed by using well established routines like nlm or optimize in the R statistical package. Our numerical calculations showed that the surface of (39) was smooth for given smooth functions $g(\cdot)$ and $G(\cdot)$. The routines were able to locate the maximum in all cases and for different starting values. However, to easy the computations it is useful to have reasonable starting values. These can be obtained, for example, by the method of moments. For $r = 1, \ldots, q+2$, let $m_r = n^{-1}\sum_{i=1}^{n} x_i^r$ denote the first $q+2$ sample moments. Equating these moments with the theoretical ones given in Eq. (16), we have $m_r = E(X^r)$, for $r = 1, \ldots, q+2$. These equations can be solved simultaneously to obtain the moments estimates.

For interval estimation of $(\alpha, \beta, \boldsymbol{\Theta})$ and hypothesis tests, we require the observed information matrix. The observed information matrix for $(\alpha, \beta, \boldsymbol{\Theta})$ can be determined as

$$\mathbf{I} = \begin{pmatrix} \mathbf{I}_{11} & \mathbf{I}_{12} & \mathbf{I}_{13} \\ \mathbf{I}_{12} & \mathbf{I}_{22} & \mathbf{I}_{23} \\ \mathbf{I}_{13} & \mathbf{I}_{23} & \mathbf{I}_{33} \end{pmatrix},$$

where

$$\mathbf{I}_{11} = \frac{\partial^2 \log L}{\partial \widehat{\alpha}^2} = -\frac{n}{\alpha^2},$$

$$\mathbf{I}_{12} = \frac{\partial^2 \log L}{\partial \widehat{\beta} \partial \widehat{\alpha}} = \frac{n}{\beta} - \sum_{i=1}^{n} \frac{1}{\beta - \log[1 - G(x_i)]}$$

$$\mathbf{I}_{13} = \frac{\partial^2 \log L}{\partial \widehat{\boldsymbol{\Theta}} \partial \widehat{\beta}} = -\sum_{i=1}^{n} \frac{\dot{G}(x_i)}{\{\beta - \log[1 - G(x_i)]\}[1 - G(x_i)]}$$

$$\mathbf{I}_{22} = \frac{\partial^2 \log L}{\partial \widehat{\beta}^2} = -\frac{\alpha n}{\beta^2} + (\alpha+1)\sum_{i=1}^{n} \frac{1}{\{\beta - \log[1 - G(x_i)]\}^2}$$

$$\mathbf{I}_{23} = \frac{\partial^2 \log L}{\partial \widehat{\boldsymbol{\Theta}} \partial \widehat{\beta}} = (\alpha+1)\sum_{i=1}^{n} \frac{\dot{G}(x_i)}{\{\beta - \log[1 - G(x_i)]\}[1 - G(x_i)]},$$

$$\begin{aligned}\mathbf{I}_{33} = \frac{\partial^2 \log L}{\partial \widehat{\boldsymbol{\Theta}}^2} &= \sum_{i=1}^{n} \frac{\ddot{g}(x_i)g(x_i) - [\dot{g}(x_i)]^2}{g^2(x_i)} \\ &- (\alpha+1)\left\{ \sum_{i=1}^{n} \frac{\ddot{G}(x_i)}{\{\beta - \log[1 - G(x_i)]\}[1 - G(x_i)]} \right. \\ &\left. - \sum_{i=1}^{n} \frac{\left[\dot{G}(x_i)\right]^2 \{1 - \beta + \log[1 - G(x_i)]\}}{\{\beta - \log[1 - G(x_i)]\}^2 [1 - G(x_i)]^2} \right\} \\ &+ \sum_{i=0}^{n} \frac{\ddot{g}(x_i)g(x_i) - [\dot{g}(x_i)]^2}{g^2(x_i)},\end{aligned}$$

where $\ddot{g}(x_i) = \partial^2 g(x_i)/\partial\theta^2$ and $\ddot{G}(x_i) = \partial^2 G(x_i)/\partial\theta^2$.

For large n, the distribution of $(\widehat{\alpha} - \alpha, \widehat{\beta} - \beta, \widehat{\boldsymbol{\Theta}} - \boldsymbol{\Theta})$ can be approximated by a $(q+2)$ multivariate normal distribution with zero means and variance–covariance matrix $\mathbf{I}^{-1}$. Some statistical properties of $(\widehat{\alpha}, \widehat{\beta}, \widehat{\boldsymbol{\Theta}})$ can be derived based on this normal approximation.

For the special case of the LE distribution with parameters λ, α and β, we obtain the log-likelihood function $\log L$, its first derivatives and the observed information matrix. Inserting the pdf and cdf of the exponential distribution in (39), the log-likelihood function for the LE model parameters becomes

$$\log L = n \log \alpha + \alpha n \log(\beta) + n \log(\lambda) - (\alpha+1)\sum_{i=1}^{n} \log(\beta + \lambda x_i).$$

The first derivatives of $\log L$ with respect to the parameters α, β and λ are given by

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} + n \log \beta - \sum_{i=1}^{n} \log(\beta + \lambda x_i),$$

$$\frac{\partial \log L}{\partial \beta} = \frac{\alpha n}{\beta} - (\alpha+1)\sum_{i=1}^{n} \frac{1}{\beta + \lambda x_i},$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - (\alpha+1)\sum_{i=1}^{n} \frac{x_i}{\beta + \lambda x_i}.$$

Based on the general results, we calculate the second derivatives of $\log L$ with respect to the parameters α, β and λ for the LE distribution. Further, we can express the observed information matrix for (α, β, λ) as

$$\mathbf{I} = \begin{pmatrix} \mathbf{I}_{11} & \mathbf{I}_{12} & \mathbf{I}_{13} \\ \mathbf{I}_{12} & \mathbf{I}_{22} & \mathbf{I}_{23} \\ \mathbf{I}_{13} & \mathbf{I}_{23} & \mathbf{I}_{33} \end{pmatrix},$$

where

$$\mathbf{I}_{11} = \frac{\partial^2 \log L}{\partial \widehat{\alpha}^2} = -\frac{n}{\alpha^2},$$
$$\mathbf{I}_{12} = \frac{\partial^2 \log L}{\partial \widehat{\beta} \partial \widehat{\alpha}} = \frac{n}{\beta} - \sum_{i=1}^{n} \frac{1}{\beta + \lambda x_i},$$
$$\mathbf{I}_{13} = \frac{\partial^2 \log L}{\partial \widehat{\lambda} \partial \widehat{\beta}} = (\alpha + 1) \sum_{i=1}^{n} \frac{x_i}{(\beta + \lambda x_i)^2},$$
$$\mathbf{I}_{22} = \frac{\partial^2 \log L}{\partial \widehat{\beta}^2} = -\frac{\alpha n}{\beta^2} + (\alpha + 1) \sum_{i=1}^{n} \frac{1}{(\beta + \lambda x_i)^2},$$
$$\mathbf{I}_{23} = \frac{\partial^2 \log L}{\partial \widehat{\alpha} \partial \widehat{\beta}} = -\sum_{i=1}^{n} \frac{x_i}{\beta + \lambda x_i},$$
$$\mathbf{I}_{33} = \frac{\partial^2 \log L}{\partial \widehat{\lambda}^2} = -\frac{n}{\lambda^2} + (\alpha + 1) \sum_{i=1}^{n} \frac{x_i^2}{(\beta + \lambda x_i)^2}.$$

6. The Lomax-exponential minification process

The first minification process having the structure

$$X_n = K \min(X_{n-1}, \varepsilon_n), \quad n \geq 1$$

was introduced by Tavares [30,31], where $K > 1$ and ε_n is an innovation sequence of independent and identically distributed exponential random variables. By means of mathematical induction, he showed that $\{X_n\}$ is the Markovian and stationary stochastic process with marginal exponential distribution too.

Several authors derived new distributions using the Marshal–Olkin method applied to the theory of minification processes. Some recent results are: the Marshall–Olkin log-logistic minification process [17], Marshall–Olkin q-Weibull distribution and max–min processes [20,21], Marshall–Olkin beta distribution and its applications (Jose et al., 2009), Marshall–Olkin gamma minification process [27] and Marshall–Olkin semi-Weibull minification process (Thomas and Jose, 2005). Cifarelli et al. [7] proposed generalized semi-Pareto and semi-Burr distributions and random coefficient minification processes. Here, we study minification process with marginal LE distribution and derive some mathematical properties.

The cdf of the LE distribution with parameters λ, α and β, say LE(λ, α, β), is given by

$$F(x) = 1 - \left(\frac{\beta}{\beta + \lambda x}\right)^{\alpha}. \tag{40}$$

We introduce a minification process with marginal LE(λ, α, β) distribution. Consider a process $\{X_n : n \geq 0\}$ defined by

$$X_n = \begin{cases} K X_{n-1}, & \text{with probability } K^{-\alpha}, \\ K \min(X_{n-1}, \varepsilon_n), & \text{with probability } 1 - K^{-\alpha}, \end{cases} \tag{41}$$

where $\{\varepsilon_n : n \geq 1\}$ is a sequence of iid random variables, X_{n-l} and ε_n are independent random variables for all $l \geq 1$ and $K > 1$.

Theorem 1. *For the minification process given by (41) with X_0 distributed as LE(λ, α, β), where $\alpha \in \mathbb{N}$, the sequence $\{X_n : n \geq 0\}$ is a stationary Markovian autoregressive process with LE(λ, α, β) marginal distribution if and only if $\{\varepsilon_n : n \geq 1\}$ follows a mixture of LE distributions given by*

$$\varepsilon_n : \begin{cases} \text{LE}(K\lambda, 1, \beta), & \text{with probability } \binom{\alpha}{1} \frac{K-1}{K^{\alpha}-1}, \\ \text{LE}(K\lambda, 2, \beta), & \text{with probability } \binom{\alpha}{2} \frac{(K-1)^2}{K^{\alpha}-1}, \\ \vdots & \\ \text{LE}(K\lambda, \alpha, \beta), & \text{with probability } \frac{(K-1)^{\alpha}}{K^{\alpha}-1}. \end{cases} \tag{42}$$

Proof. The survival function $\overline{F}(x) = P\{X > x\}$ of the model (41) is given by

$$\overline{F}_{X_n}(xK) = K^{-\alpha}\overline{F}_{X_{n-1}}(x) + (K^{\alpha}-1)K^{-\alpha}\overline{F}_{X_{n-1}}(x)\overline{F}_{\varepsilon_n}(x).$$

Using the facts that X_0 has the LE(λ,α,β) distribution, ε_1 has the mixture of the LE distributions and $\alpha \in \mathbb{N}$, we obtain

$$\begin{aligned}
\overline{F}_{X_1}(xK) &= K^{-\alpha}\left[\frac{\beta}{\beta+\lambda x}\right]^{\alpha} + \frac{K^{\alpha}-1}{K^{\alpha}}\left[\frac{\beta}{\beta+\lambda x}\right]^{\alpha}\left\{\binom{\alpha}{1}\frac{K-1}{K^{\alpha}-1}\frac{\beta}{\beta+\lambda Kx}\right.\\
&\quad\left.+\binom{\alpha}{2}\frac{(K-1)^2}{K^{\alpha}-1}\left[\frac{\beta}{\beta+\lambda Kx}\right]^2+\cdots+\binom{\alpha}{\alpha}\frac{(K-1)^{\alpha}}{K^{\alpha}-1}\left[\frac{\beta}{\beta+\lambda Kx}\right]^{\alpha}\right\}\\
&= K^{-\alpha}\left[\frac{\beta}{\beta+\lambda x}\right]^{\alpha} + \left[\frac{\beta}{\beta+\lambda x}\right]^{\alpha}\left\{\sum_{i=1}^{\alpha}\binom{\alpha}{i}\left(\frac{1}{K}\right)^{\alpha-i}\left(1-\frac{1}{K}\right)^{i}\left[\frac{\beta}{\beta+\lambda Kx}\right]^{i}\right\}\\
&= K^{-\alpha}\left[\frac{\beta}{\beta+\lambda x}\right]^{\alpha} + \left[\frac{\beta}{\beta+\lambda x}\right]^{\alpha}\left\{\left[\frac{\frac{\beta}{\beta+\lambda Kx}}{\frac{\beta}{\beta+\lambda x}}\right]^{\alpha}-\frac{1}{K^{\alpha}}\right\} = \left[\frac{\beta}{\beta+\lambda Kx}\right]^{\alpha}.
\end{aligned}$$

Thus, X_1 has the LE(λ,α,β) distribution. Using mathematical induction one can prove that X_n possesses the same distribution. So, $\{X_n : n \geq 0\}$ is a stationary Markovian autoregressive model having the LE(λ,α,β) marginal distribution.

On the other hand, let $\{X_n : n \geq 0\}$ be a stationary Markovian autoregressive model with LE(λ,α,β) marginal distribution. Then, the equation of the stationary equilibrium is given by

$$\begin{aligned}
\overline{F}_{\varepsilon}(x) &= \frac{K^{\alpha}}{K^{\alpha}-1}\left[\frac{\overline{F}_x(xK)}{\overline{F}_x(x)}-\frac{1}{K^{\alpha}}\right] = \frac{K^{\alpha}}{K^{\alpha}-1}\left[\left(\frac{\beta+\lambda x}{\beta+\lambda Kx}\right)^{\alpha}-\frac{1}{K^{\alpha}}\right]\\
&= \frac{K^{\alpha}}{K^{\alpha}-1}\left\{\left[\frac{1}{K}+\left(1-\frac{1}{K}\right)\frac{\beta}{\beta+\lambda Kx}\right]^{\alpha}-\frac{1}{K^{\alpha}}\right\}\\
&= \frac{K^{\alpha}}{K^{\alpha}-1}\left[\sum_{i=1}^{\alpha}\binom{\alpha}{i}\left(\frac{1}{K}\right)^{\alpha-i}\left(1-\frac{1}{K}\right)^{i}\left(\frac{\beta}{\beta+\lambda Kx}\right)^{i}\right],
\end{aligned}$$

i.e. ε_n has a mixture of LE distributions. We have proved Theorem 1.

Now, we derive the joint survival function of (X_n, X_{n-1}). One obtains

$$\begin{aligned}
\overline{S}(y,x) &= P\{X_n > y, X_{n-1} > x\}\\
&= K^{-\alpha}P\{KX_{n-1} > y, X_{n-1} > x\} + \left(1-\frac{1}{K^{\alpha}}\right)P\{K\min(X_{n-1},\varepsilon_n) > y, X_{n-1} > x\}\\
&= K^{-\alpha}P\left\{X_{n-1} > \max\left(\frac{y}{K},x\right)\right\} + \left(1-\frac{1}{K^{\alpha}}\right)P\left\{X_{n-1} > \max\left(\frac{y}{K},x\right)\right\}\\
&\quad\times P\left\{\varepsilon_n > \frac{y}{K}\right\}P\{X_{n-1} > x\}.
\end{aligned}$$

If $y \geqslant Kx$, we have

$$\overline{S}(y,x) = \left(\frac{\beta}{\beta K+\lambda y}\right)^{\alpha} + \frac{\beta^{2\alpha}}{(\beta K+\lambda y)^{\alpha}(\beta+\lambda x)^{\alpha}}\sum_{i=1}^{\alpha}\binom{\alpha}{i}(K-1)^{i}\left(\frac{K\beta}{\beta K+\lambda y}\right)^{i}.$$

On the other hand, for $y < Kx$, we obtain

$$\overline{S}(y,x) = \left[\frac{\beta}{K(\beta+\lambda x)}\right]^{\alpha} + \frac{\beta^{2\alpha}}{(\beta+\lambda x)^{2\alpha}K^{\alpha}}\sum_{i=1}^{\alpha}\binom{\alpha}{i}(K-1)^{i}\left(\frac{K\beta}{\beta K+\lambda y}\right)^{i}.$$

7. The Log-Lomax–Weibull regression model

If X is a random variable having the LW density function (8), $Y = \log(X)$ has the log-Lomax–Weibull (LLW) distribution. The density function of Y, parameterized in terms of $\sigma = a^{-1}$ and $\mu = -\log(b)$, can be expressed as

$$f(y;\alpha,\beta,\mu,\sigma) = \frac{\alpha\beta^{\alpha}\exp\left(\frac{y-\mu}{\sigma}\right)}{\sigma\left[\beta+\exp\left(\frac{y-\mu}{\sigma}\right)\right]^{\alpha+1}}, \quad y \in \mathbb{R}, \tag{43}$$

where $\alpha > 0, \beta > 0$ $\sigma > 0$ and $\mu \in \mathbb{R}$. Plots of the density function (43) for selected parameter values are displayed in Fig. 4. These plots show great flexibility for different values of the shape parameters α and β. If Y is a random variable having density function (43), we write $Y \sim$ LLW$(\alpha,\beta,\mu,\sigma)$.

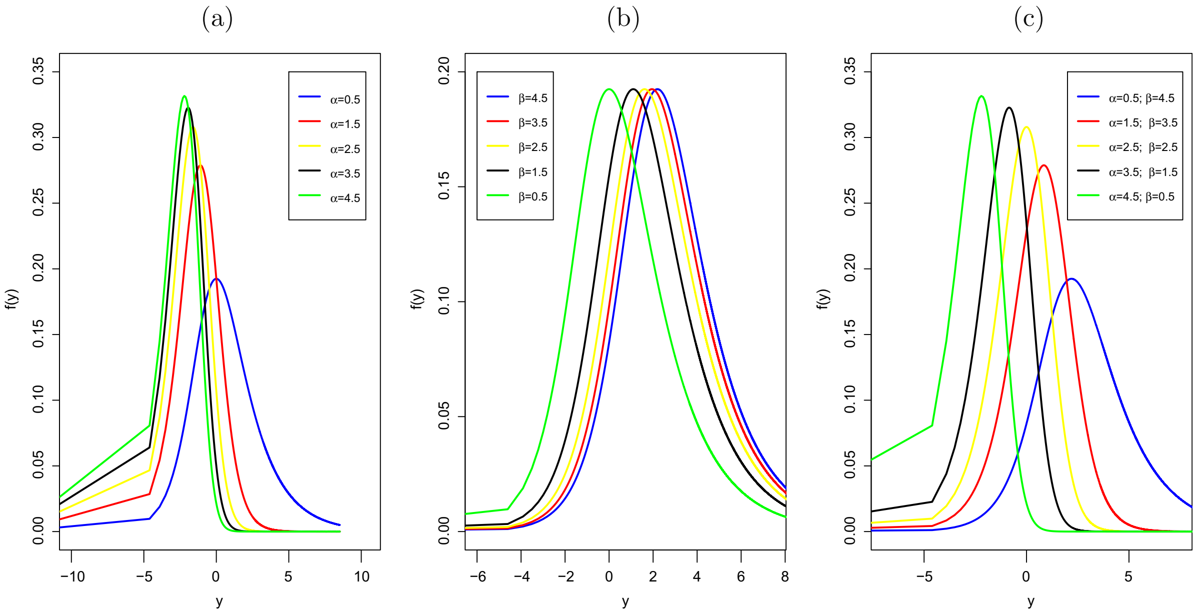

Fig. 4. Plots of the LLW densities: (a) α increasing, $\beta = 0.5$, $\mu = 0$ and $\sigma = 1$ (b) β decreasing, $\alpha = 0.5$, $\mu = 0$ and $\sigma = 1$ and (c) α increasing and β decreasing, $\mu = 0$ and $\sigma = 1$.

Thus, if $X \sim \mathrm{LW}(\alpha, \lambda, a, b)$, then $Y = \log(X) \sim \mathrm{LLW}(\alpha, \lambda, \mu, \sigma)$.

The survival function corresponding to (43) is given by

$$S(y) = \left[\frac{\beta}{\beta + \exp\left(\frac{y-\mu}{\sigma}\right)}\right]^{\alpha}.$$

We define the standardized random variable $Z = (Y - \mu)/\sigma$ with density function

$$\pi(z; \alpha, \beta) = \frac{\alpha\beta^{\alpha}\exp(z)}{[\beta + \exp(z)]^{\alpha+1}}, \quad z \in \mathbb{R}. \tag{44}$$

Based on a result in [25, Section 2.6.10, Eq. (41)], the nth ordinary moment of the standardized distribution (44) reduces to

$$E(Z^n) = \beta^{1-2(\alpha+1)} \frac{\partial^n}{\partial \rho^n} [B(\rho + 1, \alpha - \rho)\,{}_2F_1(\alpha, \alpha - \rho, \alpha + 1; 0)]_{\rho=0}.$$

In many practical applications, the lifetimes x_i are affected by explanatory variables such as the cholesterol level, blood pressure and many others. Let $\mathbf{v}_i = (v_{i1}, \ldots, v_{ip})^T$ be the explanatory variable vector associated with the ith response variable y_i, for $i = 1, \ldots, n$. Consider a sample $(y_1, \mathbf{v}_1), \ldots, (y_n, \mathbf{v}_n)$ of n independent observations, where each random response is defined by $y_i = \min\{\log(x_i), \log(c_i)\}$, and $\log(x_i)$ and $\log(c_i)$ are the log-lifetime and log-censoring, respectively. We consider non-informative censoring such that the observed lifetimes and censoring times are independent.

Now, we construct a linear regression model for the response variable y_i based on the LLW distribution given by

$$y_i = \mathbf{v}_i^T \boldsymbol{\gamma} + \sigma z_i, \quad i = 1, \ldots, n, \tag{45}$$

where the random error z_i has the density function (44), $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_p)^T, \sigma > 0$ is a scale parameter, $\alpha > 0$ and $\beta > 0$ are shape parameters and $\mathbf{v}_i$ is the vector of explanatory variables modeling the location parameter $\mu_i = \mathbf{v}_i^T \boldsymbol{\gamma}$. Hence, the location parameter vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^T$ of the LLW model has a linear structure $\boldsymbol{\mu} = \mathbf{V}\boldsymbol{\gamma}$, where $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)^T$ is a known model matrix. The log-Lomax exponential (LLE) regression model is defined by (45) with $\sigma = 1$.

Let F and C be the sets of individuals for which y_i is the log-lifetime or log-censoring, respectively. The total log-likelihood function for the model parameters $\boldsymbol{\theta} = (\alpha, \beta, \sigma, \boldsymbol{\gamma}^T)^T$ can be expressed from (44) and (45) as

$$l(\boldsymbol{\theta}) = q\,[\log(\alpha) + \alpha\log(\beta) - \log(\sigma)] + (n-q)\alpha\log(\beta) + \sum_{i \in F}\left(\frac{y_i - \mathbf{v}_i^T\boldsymbol{\gamma}}{\sigma}\right) - (\alpha+1) + \sum_{i \in F}\log\left[\beta + \exp\left(\frac{y_i - \mathbf{v}_i^T\boldsymbol{\gamma}}{\sigma}\right)\right]$$
$$- \alpha\sum_{i \in C}\log\left[\beta + \exp\left(\frac{y_i - \mathbf{v}_i^T\boldsymbol{\gamma}}{\sigma}\right)\right], \tag{46}$$

Table 2
Descriptive statistics for INPC data set.

Mean	Median	SD	Variance	Skewness	Kurtosis	Min.	Max.
0.64	0.50	0.60	0.36	1.56	6.59	−0.49	3.39

where q is the observed number of failures. The MLE $\widehat{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$ can be obtained by maximizing the log-likelihood function (46). Initial values for $\boldsymbol{\gamma}$ and σ are taken from the fit of the log-Weibull (or extreme value) regression model. From the fitted model (45), the survival function for y_i can be estimated by

$$S(y_i;\hat{\alpha},\hat{\beta},\hat{\sigma},\widehat{\boldsymbol{\gamma}}^T)=\left[\frac{\hat{\beta}}{\hat{\beta}+\exp\left(\frac{y_i-\mathbf{v}_i^T\hat{\gamma}}{\hat{\sigma}}\right)}\right]^{\hat{\alpha}}. \tag{47}$$

Under general regularity conditions, the asymptotic distribution of $(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta})$ is multivariate normal $N_{p+3}(0,K(\boldsymbol{\theta})^{-1})$, where $K(\boldsymbol{\theta})$ is the expected information matrix. The asymptotic covariance matrix $K(\boldsymbol{\theta})^{-1}$ of $\widehat{\boldsymbol{\theta}}$ can be approximated by the inverse of the $(p+3)\times(p+3)$ observed information matrix $\mathbf{I}(\widehat{\boldsymbol{\theta}})$ evaluated at $\widehat{\boldsymbol{\theta}}$. The multivariate normal $N_{p+3}(0,\mathbf{I}(\widehat{\boldsymbol{\theta}})^{-1})$ distribution can be used to construct approximate confidence regions for some parameters in $\boldsymbol{\theta}$ and for the hazard rate and survival functions. In fact, an $100(1-\alpha)\%$ asymptotic confidence interval for each parameter θ_r is given by

$$ACI_r=\left(\hat{\theta}_r-z_{\alpha/2}\sqrt{-\widehat{I}^{r,r}},\ \hat{\theta}_r+z_{\alpha/2}\sqrt{-\widehat{I}^{r,r}}\right),$$

where $-\widehat{I}^{r,r}$ represents the rth diagonal element of the inverse of the estimated observed information matrix $\mathbf{I}(\widehat{\boldsymbol{\theta}})^{-1}$ and $z_{\alpha/2}$ is the quantile $1-\alpha/2$ of the standard normal distribution. The likelihood ratio (LR) statistic can be used to discriminate between the LLW and LLE regression models since they are nested models. In this case, the hypotheses to be tested are $H_0:\sigma=1$ versus $H_1: H_0$ *is not true*, and the LR statistic reduces to $w=2\{l(\widehat{\boldsymbol{\theta}})-l(\widetilde{\boldsymbol{\theta}})\}$, where $\widetilde{\boldsymbol{\theta}}$ is the MLE of $\boldsymbol{\theta}$ under H_0. The null hypothesis is rejected if $w>\chi^2_{1-\alpha}(1)$, where $\chi^2_{1-\alpha}(1)$ is the quantile of the chi-square distribution with two degrees of freedom.

8. Applications

We illustrate the importance of the proposed family in three applications to real data. In the last few years, several extensions of the normal and Weibull distributions have been introduced in the literature. For example, Cordeiro et al. [11] and

Table 3
MLEs and the corresponding SEs (given in parentheses) of the model parameters for the INPC data and the AIC and BIC measures.

Model						AIC	BIC
LN	μ 0.6160 (0.0204)	σ 0.3619 (0.0216)	α 1.2148 (0.2565)	β 0.6641 (0.2357)		242.1	254.3
BN	μ −0.4391 (0.1590)	σ 0.4686 (0.0028)	a 5.3041 (2.4669)	b 0.2905 (0.0431)		256.0	268.3
Normal	μ 0.6442 (0.0477)	σ 0.5988 (0.0337)				288.5	294.6
GN	μ 0.3098 (0.2000)	σ 0.3771 (0.0028)	α_1 0.7881 (0.2176)	β_1 3.3616 (0.4486)		278.2	290.4
KwN	μ −0.7309 (0.0179)	σ 0.5338 (0.0152)	a 14.5106 (2.0541)	b 0.2964 (0.0338)		252.3	264.5
McN	μ −1.2530 (0.0205)	σ 0.5993 (0.0178)	a 13.9336 (0.0631)	b 0.2858 (0.0307)	c 3.8102 (0.0412)	251.1	266.4
SN	μ −0.0282 (0.0480)	σ 0.9005 (0.0622)	λ 4.3606 (0.0970)			250.0	259.1

Alzaatreh et al. [3] pioneered the McDonald normal (McN) and gamma-normal (GN) distributions, respectively, and Cordeiro et al. [9] defined the four-parameter lifetime model called the Kumaraswamy Weibull (KwW) distribution. We compare the fits of the LN and LW distributions with those of the beta normal (BN), Kumaraswamy normal (KwN) [10], McN, GN, skew-normal (SN), Weibull, beta Weibull (BW), Kumaraswamy Weibull (KwW), LE, Burr type II, exponentiated Pareto (EPa) [18], beta Pareto (BPa) [2] and their baseline distributions themselves.

8.1. INPC data

The INPC is a national index of consumer prices of Brazil, released by IBGE (Brazilian Institute of Geography and Statistics). The period of collection goes from day 1 to 30 of the reference month and the target population includes families dwelling in the urban areas, whose head of the household is considered the main employee. The survey was conducted in the metropolitan regions of Belém, Belo Horizonte, Brasília, Curitiba, Fortaleza, Goiânia, Porto Alegre, Recife, Rio de Janeiro, São Paulo and Salvador. The data set was extracted from IBGE database available at http://www.ibge.gov.br. Table 2

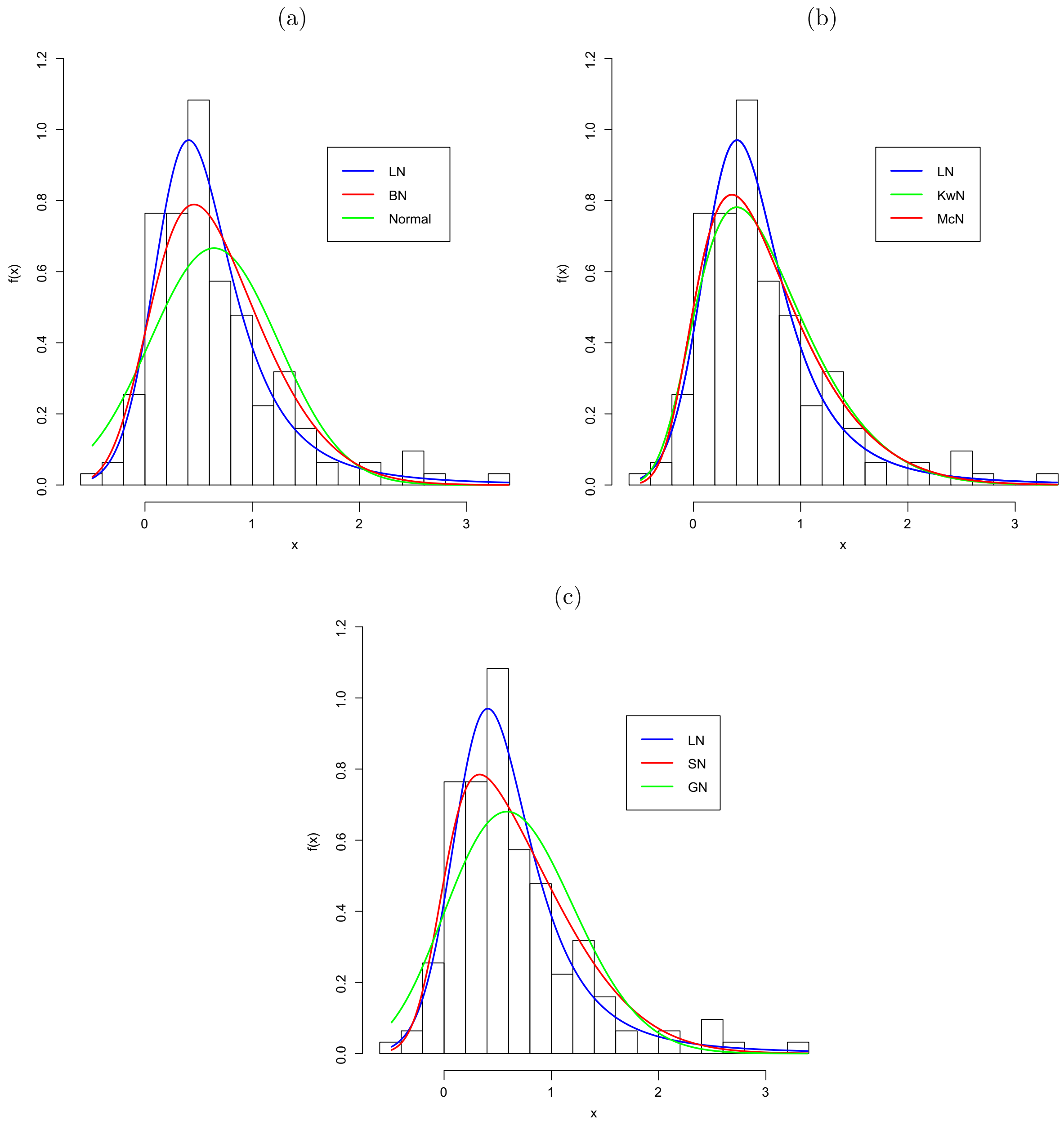

Fig. 5. (a) Estimated densities of the LN, BN and normal models. (b) Estimated densities of the LN, KwN and McN models. (c) Estimated densities of the LN, SN and GN models.

Table 4
Formal goodness-of-fit tests for the INPC data.

Model	Statistic	
	W^*	A^*
LN	0.0329	0.2335
BN	0.2465	1.5050
Normal	0.7635	4.4915
GN	0.5742	3.4008
KwN	0.1932	1.1928
McN	0.1401	0.8813
SN	0.1800	1.1148

Table 5
Descriptive statistics for lifetime data.

Mean	Median	SD	Variance	Skewness	Kurtosis	Min.	Max.
99.82	70.0	81.12	6580.12	1.84	2.89	12.0	376.0

Table 6
MLEs and the corresponding SEs (given in parentheses) of the model parameters for the lifetime data and the measures AIC and BIC.

Model					AIC	BIC
LW	a	b	α	β	786.9	796.0
	3.0207	0.0166	0.6701	1.0252		
	(0.5879)	(0.0114)	(0.2439)	(2.1767)		
BW	a	b	λ	φ	788.1	797.2
	0.4977	0.1948	21.7663	0.7808		
	(0.1797)	(0.1649)	(3.4277)	(0.5887)		
KwW	a	b	λ_1	φ_1	788.2	797.3
	0.4777	0.1868	21.7973	0.9193		
	(0.2000)	(0.0028)	(0.2176)	(0.4486)		
LE	b	α	β		814.8	821.6
	0.0200	13.9764	26.8142			
	(0.0336)	(9.3243)	(47.9825)			
BPa	v	θ	λ_2		787.4	794.2
	0.3299	42.1434	13.5918			
	(0.1988)	(10.7091)	(12.5797)			
EPa	v	θ			797.5	802.1
	1.2482	150.1257				
	(0.0884)	(50.7663)				
Burr	c	k			985.1	989.6
	3.1674	0.0726				
	(9.6311)	(0.2210)				
Weibull	a	b			798.2	802.8
	1.3915	0.0090				
	(0.1180)	(0.0008)				

presents a descriptive summary for the INPC data which suggests skewed distributions with high degrees of skewness and kurtosis. We compare the fits of the LN distribution with the normal, BN, KwN, McN, GN and SN distributions to these data.

Table 3 gives the MLEs and the corresponding SEs (in parentheses) of the model parameters and the values of the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) for the fitted models. The computations are performed using the statistical software R. The AIC and BIC values for the LN model are the smallest values among those ones of the other fitted models.

Fig. 5 displays the histogram of the data and the fitted LN density function and some densities of non-nested models, respectively. We note that the LN distribution produces better fit than the other models.

We also apply formal goodness-of-fit tests in order to verify which distribution fits the data better. We consider the Cramér-Von Mises (W^*) and Anderson–Darling (A^*) statistics. In general, the smaller the values of the statistics, W^* and A^*, the better the fit to the data. For further details, the reader is referred to [8]. The values of the statistics

W^* and A^* for the distributions are listed in Table 4. Overall, by comparing the measures of these formal goodness-of-fit tests in Table 4, we conclude that the LN distribution yields a better fit than the normal, BN, KwN, GN, McN and SN distributions and therefore can be an interesting alternative to these distributions for modeling skewed data. These results illustrate the potentiality of the new model and the importance of the two extra shape parameters in the LG family.

8.2. Lifetime data

We consider the lifetimes corresponding to 72 survival times of guinea pigs injected with different doses of tubercle bacilli. These data were analyzed by Kundu et al. [22]. Table 5 gives a descriptive summary of the data and suggests skewed distributions. The LW model seems to account very well for the degrees of skewness and kurtosis presented in the current data.

Table 6 lists the MLEs and their standard errors (in parentheses) of the parameters from the fitted LW, BW, KwW, LE, Burr type II, EPa, BPa and Weibull models and the values of the statistics AIC and BIC. The computations were performed using the

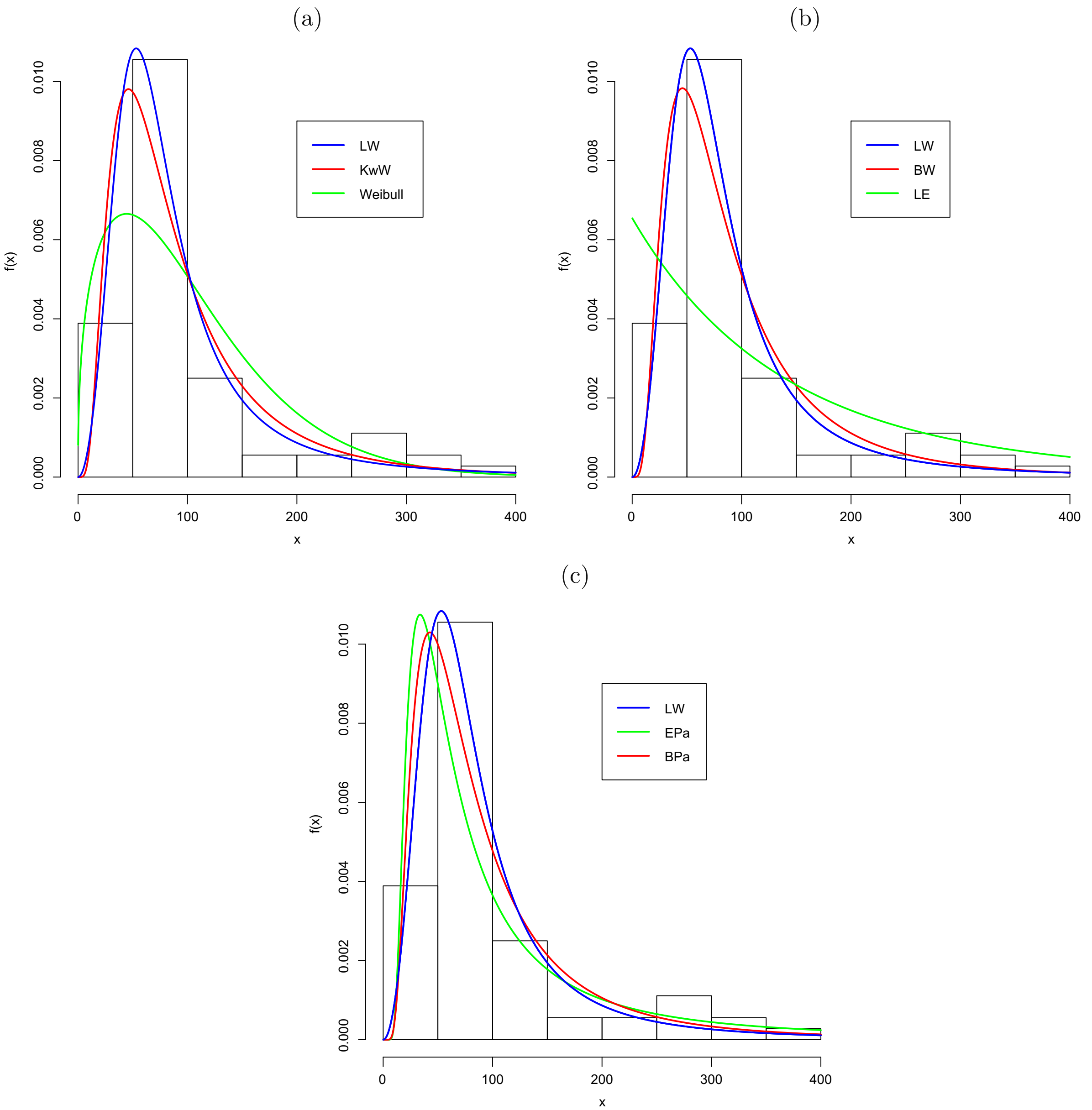

Fig. 6. (a) Estimated densities of the LW, KwW and Weibull models. (b) Estimated densities of the LW, BW and LE models. (c) Estimated densities of the LW, EPa and BPa models.

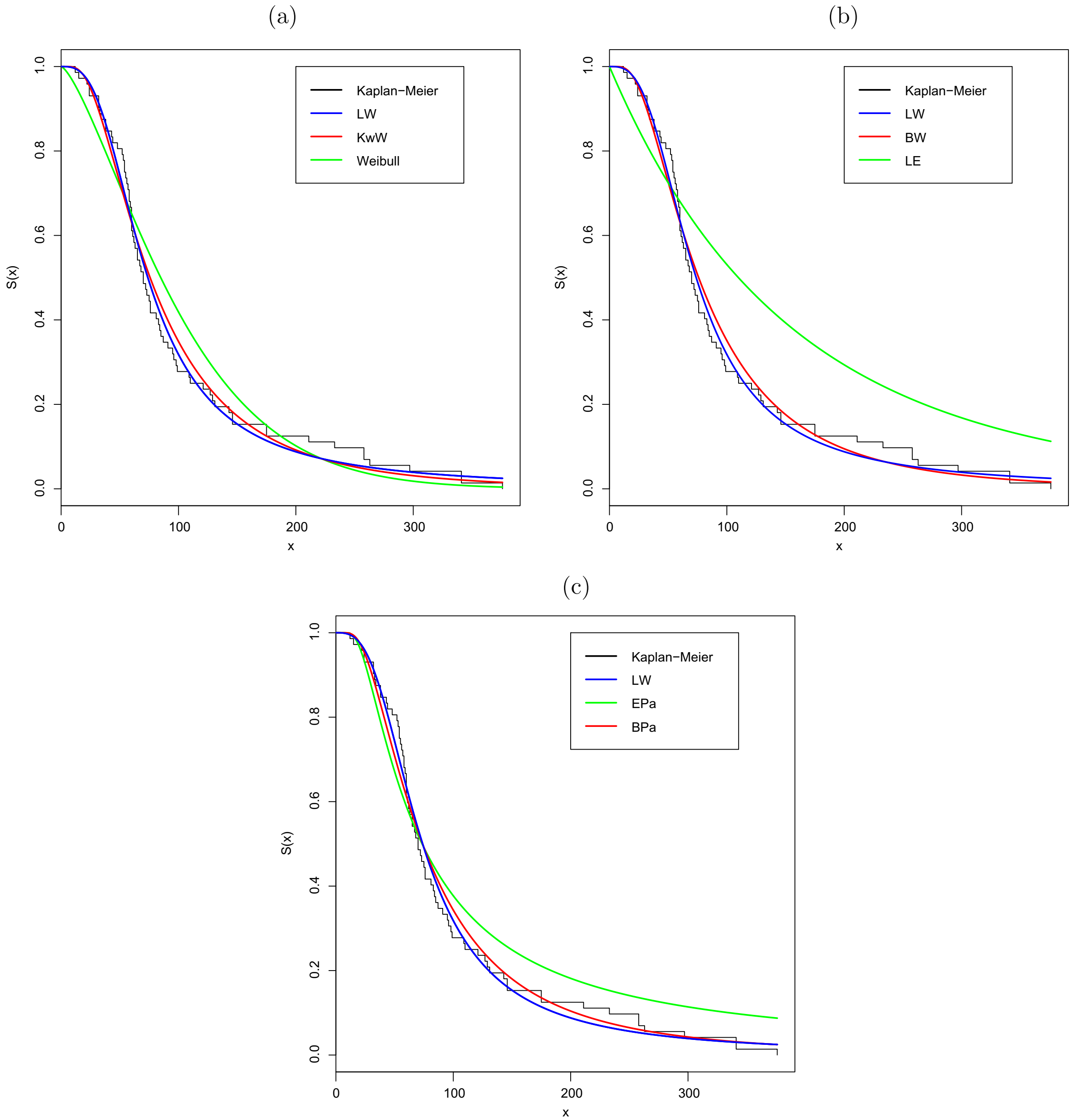

Fig. 7. (a) Empirical and estimated survival functions of the LW, KwW and Weibull models. (b) Empirical and estimated survival functions of the LW, BW and LE models. (c) Empirical and estimated survival functions of the LW, EPa and BPa models.

statistical software R. The results indicate that the LW model has the smallest values of these statistics among the fitted models, and therefore it could be chosen as the best model.

In order to assess if the new model is appropriate, Fig. 6(a)–(c) display the histogram of the data and the fitted LW density function and some densities of well-known models. Further, Fig. 7(a)–(c) display plots of the empirical and estimated survival functions of the LW distribution and other models. We can conclude that the LW distribution is a very suitable model to fit the current data.

We apply formal goodness-of-fit tests based on the statistics W^* and A^* as defined before in order to verify which distribution provides a better to these data. The values of these statistics for the fitted models are listed in Table 7. Overall, by comparing the measures of these formal goodness-of-fit tests in Table 7, we conclude that the LW distribution yields a better fit than the Weibull, BW, KwW, LE, BPa, EPa and Burr distributions and therefore it can be an interesting alternative to these distributions for modeling lifetime data. These results illustrate the importance of the LW distribution.

Table 7
Formal goodness-of-fit tests for the lifetime data.

Model	Statistic	
	W^*	A^*
LW	0.0787	0.4724
BW	0.1364	0.7484
KwW	0.1392	0.7616
LE	0.3020	1.6498
BPa	0.1316	0.7419
EPa	0.1749	1.0423
Burr	0.1347	0.7573
Weibull	0.4343	2.3916

Table 8
MLEs of the parameters from some fitted regression models to the to class-H insulation life data set, the corresponding SEs (given in parentheses), p-value in $[\cdot]$ and the AIC and BIC measures.

Model	α	β	σ	γ_0	γ_1	AIC	BIC
LLW	0.4837	0.5578	0.1025	15.1337	−0.03259	13.8	22.2
	(0.2281)	(0.0613)	(0.0280)	(0.3338)	(0.0016)		
				[<0.0001]	[<0.0001]		
	a	b	σ	γ_0	γ_1	AIC	BIC
LBW	13.0544	1.2110	0.7738	13.8269	−0.0298	18.7	27.2
	(4.3069)	(0.4013)	(0.3033)	(0.5974)	(0.0021)		
				[<0.0001]	[<0.0001]		
Log-Weibull	1	1	0.2720	14.3024	−0.0279	22.4	27.5
	(–)	(–)	(0.0315)	(0.3176)	(0.0014)		
				[<0.0001]	[<0.0001]		

8.3. The LLW regression model

As an application of the LLW regression model, we consider the data set given in [23,p. 115], concerning "hours to failure of motorettes with a new Class-H insulation". An experiment has been designed in order to evaluate the effect of the temperature on the failure times. Four test temperatures were considered: $190, 220, 240$ and $260\,^\circ$C, and 10 motorettes were randomly assigned to each test temperature. The motorettes were periodically examined for insulation failure. The failure time (in hours) of observation i, t_i, was defined as the midway between the inspection time when the failure was found and the time of the previous inspection, and v_{i1} is the temperature (for other details, see Nelson, 2004).

We fit the LLW regression model

$$y_i = \gamma_0 + \gamma_1 v_{i1} + \sigma z_i, \tag{48}$$

where the errors $z_1, \ldots, z_{40}$ are independent random variables with density function (44).

We compare the fits of the LLW, log-beta Weibull (LBW) and log-Weibull distributions to the class-H insulation life data. The LBW density function is given by

$$f(y; a, b, \sigma, \mu) = \frac{1}{\sigma B(a, b)} \exp\left\{\left(\frac{y-\mu}{\sigma}\right) - b\exp\left(\frac{y-\mu}{\sigma}\right)\right\}\left\{1 - \exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right\}^{a-1}, \tag{49}$$

where $y \in \mathbb{R}, \sigma > 0$ and $\mu \in \mathbb{R}$. For more details, see, for example, [12,13].

Table 8 provides the MLEs of the parameters for the LLW, LBW and log-Weibull regression models fitted to the current data using the procedure NLMixed in SAS. These results indicate that the LLW model has the lowest AIC and BIC values among those of the fitted models. The values of these statistics indicate that the LLW model provides the best fit to the data. Further, we note from the fitted LLW regression model that v_1 is significant at 1% and that there is a significant difference between the temperatures levels $190, 220, 240$ and 260 for the failure times.

9. Conclusions and future work

We propose a new class of models called the Lomax-G ("LG" for short) family of distributions which can generalize all classical continuous distributions. For any parent continuous distribution G, we define the corresponding LG distribution with two extra positive parameters. So, the new class extends several common distributions such as the normal, log-normal, gamma, Weibull, Gumbel and log-logistic distributions. Some mathematical properties of the new generator such as ordinary, incomplete and factorial moments, cumulants, generating and quantile functions, mean deviations, Bonferroni and Lorenz curves, asymptotic distribution of the extreme values, Shannon entropy, Rényi entropy, reliability and order statistics are obtained in generality. The model parameters are estimated by maximum likelihood. Three examples to real data illustrate the importance and potentiality of the new family.

Properties and extensions of the family not considered in this paper are: stochastic orderings, cumulative residual entropy, Song's measure, acceptance sampling plans, goodness of fit tests, tolerance intervals, distributions of the sum, product and ratio of LG random variables, multivariate generalizations, Bayesian and empirical Bayes estimation, uniform minimum variance unbiased estimation, estimation using weighted least squares, estimation using bootstrap, estimation using quantiles, estimation using order statistics, estimation using L-moments, and estimation using record statistics. We hope to address some of these in a future paper.

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